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The Behavior of the Derivatives of the Algebraic Polynomials of Best Approximation

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1. INTRODUCTION

In a recent paper M. Hasson [3] has investigated the behavior of the derivatives of the polynomials of best approximation of a function f on $[-1, 1]$. These investigations have led to norm estimates on the derivatives of the polynomials and on the distance between these derivatives and the respective derivatives of the function. Separate results were obtained when the norms were taken over the whole interval and when they were taken on a subinterval (where the estimates are significantly better). This calls for pointwise estimates in the spirit of the results of Timan [6] and Trigub [7]. In Section 2 we obtain pointwise estimates on the distance between the derivatives of the polynomials of best approximation of f and the respective derivatives of f. These results extend and unify those of Hasson [3]. In Section 3 we consider higher-order derivatives and obtain estimates on the growth of the sequence of derivatives of the polynomials of best approximation. Some special cases of these results are due to Hasson [3].

2. SIMULTANEOUS APPROXIMATION BY THE ALGEBRAIC POLYNOMIALS OF BEST APPROXIMATION

Let $E_n(f)$ denote the rate of approximation to f by polynomials of degree $\leqslant n$ in the sup-norm on $[-1, 1]$, i.e.,

$$E_n(f) = \inf_{p \in \Pi_n} \|f - p\|,$$

where Π_n is the set of all algebraic polynomials of degree $\leqslant n$ and $\|f-p\| = \max_{-1 \leqslant x \leqslant 1} |f(x) - p(x)|$.

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Our main result yields pointwise simultaneous approximation of a function and its derivatives by its polynomials of best approximation and their derivatives.

THEOREM 1. *For* $r \geqslant 0$ *let* $f \in C^r[-1, 1]$ (f *is* r *times continuously differentiable in* $[-1, 1]$) *and let* $P_n \in \Pi_n$ *denote its polynomial (of degree* $\leqslant n$) *of best approximation on* $[-1, 1]$. *Then for each* $0 \leqslant k \leqslant r$ *and every* $-1 \leqslant x \leqslant 1$,

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leqslant \frac{C_r}{n^k} [\Delta_n(x)]^{-k} E_{n-k}(f^{(k)}), \qquad n \geqslant k, \tag{1}$$

where $\Delta_n(x) = \sqrt{1-x^2}/n + 1/n^2$ *and* C_r *is an absolute constant which depends only on* r.

Since it is well known (see [2, p. 39]) that there is an absolute constant K such that if $f \in C^1[-1, 1]$, then

$$E_n(f) \leqslant (K/n) E_{n-1}(f'), \tag{2}$$

we immediately conclude

COROLLARY 1. *Let* $f \in C^r[-1, 1]$ *and let* P_n *be its* nth *polynomial of best approximation on* $[-1, 1]$. *Then for each* $0 \leqslant k \leqslant r$ *and all* $-1 \leqslant x \leqslant T$,

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leqslant \frac{K_r}{n^r} [\Delta_n(x)]^{-k} E_{n-r}(f^{(r)}), \qquad n \geqslant r. \tag{3}$$

Other consequences of our theorem are Hasson's [3, Theorems 2.3, 2.4 and 2.8].

COROLLARY 2. *Let* $f \in C^r[-1, 1]$ *and let* P_n *be its* nth *polynomial of best approximation on* $[-1, 1]$. *Then for each* $0 \leqslant k \leqslant r$,

(i)
$$\begin{aligned} \|f^{(k)} - P_n^{(k)}\| &\leqslant C_r n^k E_{n-k}(f^{(k)}), && n \geqslant k, \\ &\leqslant K_r n^{2k-r} E_{n-r}(f^{(r)}), && n \geqslant r. \end{aligned}$$

(ii) *If* $-1 < \alpha < \beta < 1$ *and* $\|f - p\|_{[\alpha,\beta]} = \max_{\alpha \leqslant x \leqslant \beta} |f(x) - p(x)|$, *then*

$$\begin{aligned} \|f^{(k)} - P_n^{(k)}\|_{[\alpha,\beta]} &\leqslant A_r E_{n-k}(f^{(k)}), && n \geqslant k, \\ &\leqslant B_r n^{k-r} E_{n-r}(f^{(r)}), && n \geqslant r, \end{aligned}$$

where A_r *and* B_r *depend only on* α, β *and* r.

It should be noted that Theorem 1 improves upon Hasson's results—Corollary 2—only in that it demonstrates the way the constants A_r and B_r depend on the interval $[\alpha, \beta]$.

Proof of Theorem 1. Since Theorem 1 is evident for $r = 0$ we proceed by induction assuming that it is true for some $r \geqslant 0$ and that $f \in C^{r+1}[-1, 1]$. Then $f' \in C^r[-1, 1]$, so that it follows from (1) that for $0 \leqslant k \leqslant r$ and $-1 \leqslant x \leqslant 1$,

$$|f^{(k+1)}(x) - q_{n-1}^{(k)}(x)| \leqslant \frac{C_r}{n^k} [\Delta_n(x)]^{-k} E_{n-1-k}(f^{(k+1)}), \tag{4}$$

where q_{n-1} is the polynomial of degree $\leqslant n-1$ of bast approximation to f'. Let

$$F(x) = f(x) - \int_{-1}^{x} q_{n-1}(t)\, dt$$
$$= f(x) - Q_n(x),$$

then by (4)

$$|F'(x)| \leqslant C_r E_{n-1}(f'). \tag{5}$$

It is well known that (5) implies the existence of a polynomial p_n of degree $\leqslant n$ such that

$$\|F - p_n\| \leqslant \frac{C_r}{n} E_{n-1}(f') \tag{6}$$

and

$$\|p_n'\| \leqslant C_r E_{n-1}(f') \tag{7}$$

($\| \ \|$ is the sup-norm on $[-1, 1]$). In fact the Jackson operator provides such a p_n. Now, by virtue of an inequality of Timan [6, 4.8(51)], we get from (7)

$$|p_n^{(k)}(x)| \leqslant [\Delta_n(x)]^{1-k} M_r E_{n-1}(f'), \qquad 1 \leqslant k \leqslant r+1. \tag{8}$$

Let R_n denote the polynomial of degree $\leqslant n$ of best approximation to F. Then, again by Timan's inequality for $0 \leqslant k \leqslant r+1$,

$$|R_n^{(k)}(x) - p_n^{(k)}(x)| \leqslant [\Delta_n(x)]^{-k} M_r' \|R_n - p_n\| \qquad \text{(by (6))}$$
$$\leqslant [\Delta_n(x)]^{-k} M_r' \left[E_n(F) + \frac{C_r}{n} E_{n-1}(f')\right] \qquad \text{(by (2))}$$
$$\leqslant [\Delta_n(x)]^{-k} \frac{M_r''}{n} E_{n-1}(f'), \tag{9}$$

since $E_n(F) = E_n(f)$. Combining (8) and (9) we have, for $1 \leqslant k \leqslant r+1$,

$$|R_n^{(k)}(x)| \leqslant \frac{D_r}{n} [\Delta_n(x)]^{-k} E_{n-1}(f')$$

$$\leqslant \frac{D'_r}{n^k} [\Delta_n(x)]^{-k} E_{n-k}(f^{(k)}). \tag{10}$$

Now $P_n = Q_n + R_n$ is evidently the polynomial of best approximation of f and (1) for $1 \leqslant k \leqslant r+1$ follows from (4) and (10). For $k=0$ (1) is self-evident.

It is interesting to compare (3) with the estimates on simultaneous approximation of Trigub [7]. The comparison should be made with the following result that follows easily from that of Trigub. We wish to thank the referee for suggesting the short proof that follows; our original proof was somewhat cumbersome.

THEOREM 2. *For $r \geqslant 0$ let $f \in C^r[-1, 1]$ and let $n \geqslant r$. Then there exists a polynomial p_n of degree $\leqslant n$ such that*

$$|f^{(k)}(x) - p_n^{(k)}(x)| \leqslant C_r [\Delta_n(x)]^{r-k} E_{n-r}(f^{(r)}) \tag{11}$$

for $k = 0, 1, \ldots, r$ and $-1 \leqslant x \leqslant 1$. The constant C_r is independent of f, n and x but may depend on r.

Proof. Choose a polynomial q_n such that

$$\|f^{(r)} - q_n^{(r)}\| \leqslant E_{n-r}(f^{(r)}).$$

By Trigub's theorem [7] there exists a polynomial R_n such that

$$|(f-q_n)^{(k)}(x) - R_n^{(k)}(x)| \leqslant C'_r [\Delta_n(x)]^{r-k} \omega\left((f-q_n)^{(r)}, \frac{1}{n}\right)$$

$$\leqslant C_r [\Delta_n(x)]^{r-k} E_{n-r}(f^{(r)}).$$

Now take $p_n = q_n + R_n$ and the theorem follows.

Comparing (3) and (11) we see that the polynomials of best approximation in general do not provide the rate of the best simultaneous approximation near the end points.

3. HIGH-ORDER DERIVATIVES OF THE POLYNOMIALS OF BEST APPROXIMATION

In this section we are going to establish estimates on the growth of the sequence $\{P_n^{(k)}(x)\}$ $(n \geqslant 1)$ for each k larger than the order of differentiability of f, thus complementing the results in Section 2.

Our first result is

THEOREM 3. *For $r \geqslant 0$ let $f \in C^r[-1, 1]$ and let $P_n \in \Pi_n$ denote its nth polynomial of best approximation. Then for each $k > r$ there exists a constant K, depending only on k, such that for every $-1 \leqslant x \leqslant 1$*

$$|P_n^{(k)}(x)| \leqslant \frac{K}{n^r} [\Delta_n(x)]^{-k} \omega\left(f^{(r)}, \frac{1}{n}\right), \qquad n = 1, 2, \ldots, \tag{12}$$

where, as usual, $\omega(f^{(r)},\ )$ denotes the modulus of continuity of $f^{(r)}$.

As special cases we get an improvement of Hasson's [3, Theorems 3.2 and 3.3]. (In fact we improve Theorem 3.3 in that we do not have to restrict ourselves to functions whose derivatives are in Lip ε, thus closing a gap between Theorem 3.2 and 3.3. Also our constants are independent of f.)

COROLLARY 3. *Let $f \in C^r(-1, 1]$ and let P_n be its nth polynomial of best approximation. Then for each $k > r$*

(i) *There exists a constant K depending only on k such that*

$$\|P_n^{(k)}\| \leqslant K n^{2k-r} \omega\left(f^{(r)}, \frac{1}{n}\right), \qquad n = 1, 2, \ldots.$$

(ii) *If $-1 < \alpha < \beta < 1$, then there exists a constant M depending on k and on α, β such that*

$$\|P_n^{(k)}\|_{[\alpha, \beta]} \leqslant M n^{k-r} \omega\left(f^{(r)}, \frac{1}{n}\right), \qquad n = 1, 2, \ldots.$$

For the proof we need the following result of Runck [5, III Satz].

THEOREM R. *Let $f \in C^r[-1, 1]$. Then for every $n \geqslant r$ there exists a polynomial p_n of degree $\leqslant n$ such that*

$$|f^{(i)}(x) - p_n^{(i)}(x)| \leqslant C_i [\Delta_n(x)]^{r-i} \omega(f^{(r)}, \Delta_n(x)), \qquad i = 0, \ldots, r \tag{13}$$

and

$$|p_n^{(k)}(x)| \leqslant C_{r+1}[\Delta_n(x)]^{r-k}\,\omega(f^{(r)}, \Delta_n(x)), \qquad k \geqslant r+1, \tag{14}$$

C_i, $i = 0,..., r+1$ *are constants independent of* f.

Proof of Theorem 3. For $n \geqslant r$ let p_n be the polynomial of Theorem R. Then

$$|P_n^{(k)}(x)| \leqslant |P_n^{(k)}(x) - p_n^{(k)}(x)| + |p_n^{(k)}(x)|.$$

Now, by [6, 4.8(51)],

$$|P_n^{(k)}(x) - p_n^{(k)}(x)| \leqslant C_k[\Delta_n(x)]^{-k}\,\|P_n - p_n\|$$
$$\leqslant \frac{C_k}{n^r}\,[\Delta_n(x)]^{-k}\,\omega\left(f^{(r)}, \frac{1}{n}\right)$$

by virtue of (13) and the fact that P_n is the polynomial of best approximation of f. Hence by (14)

$$|P_n^{(k)}(x)| \leqslant \frac{K_k}{n^r}\,[\Delta_n(x)]^{-k}\,\omega\left(f^{(r)}, \frac{1}{n}\right).$$

Since (12) is trivial for $1 \leqslant n \leqslant r$ (the left-hand side of (12) being zero for these n) the proof of Theorem 2 is complete.

We do not know if Theorem 3 is sharp for $r = 0$ and $k = 1$ in the sense that for some function f inequality (12) can be reversed for some $-1 \leqslant x \leqslant 1$. However, Hasson [4, pp. 492–493] shows that for any f in the Zygmund class of $[-1, 1]$ if $-1 < \alpha < \beta < 1$, then

$$\|P_n''\|_{[\alpha,\beta]} \leqslant Mn. \tag{15}$$

Thus (12) is not sharp for $r = 0$ and $k \geqslant 2$.

We propose to strengthen Theorem 3 in the following way.

THEOREM 4. *Let* $f \in C[-1, 1]$ *and let* $P_n \in \Pi_n$ *denote its* nth *polynomial of best approximation. Given* $r \geqslant 1$, *for each* $k \geqslant r$ *there exists a constant* K *depending only on* k *and* r *such that for every* $-1 \leqslant x \leqslant 1$,

$$|P_n^{(k)}(x)| \leqslant K[\Delta_n(x)]^{-k}\,\omega_r\left(f, \frac{1}{n}\right), \qquad n = 1, 2,.... \tag{16}$$

Note that (15) follows immediately from (16) for $k = r = 2$.

The proof of Theorem 4 requires a lemma.

LEMMA. *Let $f \in C[-1,1]$ and let $r \geqslant 1$. Then for every $n \geqslant 1$ there exists an* nth *degree polynomial p_n such that*

$$\|f-p_n\| C\omega_r\left(f, \frac{1}{n}\right), \qquad n \geqslant 1, \tag{17}$$

$$\|p_n^{(r)}\| \leqslant Cn^r\omega_r\left(f, \frac{1}{n}\right), \qquad n \geqslant 1. \tag{18}$$

C is a constant which depends on r only.

Proof. First observe that if $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_\infty[-1,1]$, then (17) and (18) take the form

$$\|f-p_n\| \leqslant C\|f^{(r)}\|_\infty n^{-r}, \qquad n \geqslant 1, \tag{19}$$

$$\|p_n^{(r)}\| \leqslant C\|f^{(r)}\|_\infty, \qquad n \geqslant 1. \tag{20}$$

That for such an f there are polynomials p_n for which (19) and (20) hold follows immediately from Theorem R (with $r-1$ instead of r).

DeVore [1, Theorem 2.1) showed the equivalence of the K functional and the rth modulus of smoothness. Hence given $f \in C[-1,1]$ for every $0 \leqslant h \leqslant 1$ there exists a function $f_h \in W_{r,\infty}[-1,1]$, i.e., $f_h^{(r-1)}$ is absolutely continuous on $[-1,1]$ and $f_h^{(r)} \in L_\infty[-1,1]$, such that

$$\|f-f_h\| \leqslant A\omega_r(f,h), \tag{21}$$

$$\|f_h^{(r)}\|_\infty \leqslant Ah^{-r}\omega_r(f,h). \tag{22}$$

A depends on r only.

Now for $n \geqslant 1$ choose $h = 1/n$. Then by (19), (20) and (22) there is a polynomial p_n such that

$$\|f_{1/n} - p_n\| \leqslant K\omega_r\left(f, \frac{1}{n}\right), \qquad n \geqslant 1,$$

and

$$\|p_n^{(r)}\| \leqslant Kn^r\omega_r\left(f, \frac{1}{n}\right), \qquad n \geqslant 1.$$

Combining with (21) completes the proof of (17) and (18).

Proof of Theorem 4. Let p_n be the polynomial satisfying (17) and (18). Then

$$|P_n^{(k)}(x)| \leqslant |P_n^{(k)}(x) - p_n^{(k)}(x)| + |p_n^{(k)}(x)|.$$

By (18) and [6, 4.8(51)] for $k \geqslant r$,

$$|p_n^{(k)}(x)| \leqslant Kn^r[\Delta_n(x)]^{r-k}\,\omega_r\left(f, \frac{1}{n}\right), \qquad n \geqslant 1, \tag{23}$$

where K depends only on k. Also by [6, 4.8(51)] and (17),

$$\begin{aligned} |P_n^{(k)}(x) - p_n^{(k)}(x)| &\leqslant K[\Delta_n(x)]^{-k}\,\|P_n - p_n\| \\ &\leqslant K'[\Delta_n(x)]^{-k}\left[E_n(f) + \omega_r\left(f, \frac{1}{n}\right)\right] \\ &\leqslant K''[\Delta_n(x)]^{-k}\,\omega_r\left(f, \frac{1}{n}\right). \end{aligned} \tag{24}$$

Combining (23) and (24) we get (16).

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